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A Non-commutative Random Stopping Theorem

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A theory of non-commutative stopping time is presented in the case where the underlying Von Neumann algebra possesses only a faithful normal state. In particular we prove an analogue of Doob's optional stopping theorem and in the special case of the quasi-free representation of the CAR we are also able to prove the random stopping theorem. These results thus extend those established by C. Barnett and T. J. Lyons (1985, *Math. Proc. Cambridge Philos. Soc.*) to include certain type III factors. © 1990 Academic Press, Inc.

1. Notations and Preliminaries

Let $\mathscr{K}$ be a complex Hilbert space, $B(\mathscr{K})$ the bounded linear operators on $\mathscr{K}$, and $\mathscr{A} \subseteq B(\mathscr{K})$ a Von Neumann algebra with a faithful normal state w. For each non-negative real t let $\mathscr{A}_t$ be a Von Neumann subalgebra of $\mathscr{A}$, and suppose the family $\{\mathscr{A}_t : t \in \mathbb{R}^+\}$ satisfies:

(i) if $t_1, t_2 \in \mathbb{R}^+$ and $t_1 \leqslant t_2$ then $\mathscr{A}_{t_1}$ is a Von Neumann subalgebra of $\mathscr{A}_{t_2}$,

(ii) the Von Neumann algebra $\mathscr{A}$ is generated by $\bigcup_{t \geqslant 0} \mathscr{A}_t$,

(iii) $\bigcap_{t > s} \mathscr{A}_t = \mathscr{A}_s$.

Finally, suppose there exists a family $\{M_t : t \in \mathbb{R}^+\}$ of conditional expectations $M_t : \mathscr{A} \to \mathscr{A}_t$ such that

(i) $w \circ M_t = \omega \ \forall t \in \mathbb{R}^+$,

(ii) $M_t(A) = A$ if $A \in \mathscr{A}_t$,

(iii) $M_t(AXB) = AM_t(X)B \quad \forall A, B \in \mathscr{A}_t,\ X \in \mathscr{A}$.

Since w is faithful we may, without loss of generality, assume that $\mathscr{A}$ and $\mathscr{A}_t$ act in its GNS spaces $\mathscr{H}$ and $\mathscr{H}_t$, where $(\mathscr{H}, \pi, \Omega)$ and $(\mathscr{H}_t, \pi, \Omega)$ are their respective GNS triples.

For each $t \in \mathbb{R}^+$, let P_t denote the orthogonal projection of $\mathscr{H}$ onto the subspace generated by $\mathscr{A}_t \Omega$.

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PROPOSITION 1.1 [3]. *For $A \in \mathscr{A}$, we have $P_t A\Omega = M_t(A)\Omega$.*

PROPOSITION 1.2. *P_t lies in the commutant of $\mathscr{A}_t$.*

Proof. Let $B \in \mathscr{A}_t$, $A \in \mathscr{A}$. Then

$$\begin{aligned} P_t BA\Omega &= M_t(BA)\Omega \\ &= BM_t(A)\Omega \\ &= BP_t A\Omega. \end{aligned}$$

Since $\mathscr{A}\Omega$ is dense in $\mathscr{H}$, the result follows.

2. STOPPING TIME

DEFINITION 2.1. An $\mathscr{A}$-valued martingale with respect to the filtration $\{\mathscr{A}_t : t \in \mathbb{R}^+\}$ is a family $\{A_t : t \in \mathbb{R}^+\}$ with $A_t \in \mathscr{A}_t$ for each t and $M_s(A_t) = A_s$ if $s \in [0, t]$. Likewise an $\mathscr{H}$-valued martingale with respect to the filtration $\{\mathscr{H}_t : t \in \mathbb{R}^+\}$ is a family $\{\eta_t : t \in \mathbb{R}^+\}$ with $\eta_t \in \mathscr{H}_t$ and $P_s \eta_t = \eta_s$ if $s \in [0, t]$. An $\mathscr{H}$-valued martingale is called simple if it is of the form $\eta_t = P_t \eta$ for some $\eta \in \mathscr{H}$.

Remark. From now on we shall use round brackets to denote a family and understand the index to run over non-negative reals. Thus (η_t) will denote an $\mathscr{H}$-valued martingale.

DEFINITION 2.2. (i) A stopping time, τ, is an increasing family of projections (Q_t) such that $\tau(t) = Q_t \in \mathscr{A}_t$, $\tau(0) = 0$, and $\tau(\infty) = I$ [2].

Let $\mathscr{P}$ denote the set of all finite partitions of $[0, \infty]$. Then for $T \in \mathscr{P}$, say $T = \{t_0, ..., t_n\}$, we define an operator $P_{\tau(T)}$ on $\mathscr{H}$ as

$$P_{\tau(T)} = \sum_{i=1}^{n} (Q_{t_i} - Q_{t_{i-1}}) P_{t_i} = \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i}.$$

(ii) If $\tau = (Q_t)$ and $\sigma = (Q'_t)$ are two stopping times, then we say $\sigma \geqslant \tau$ iff $Q'_t \leqslant Q_t$ for each t.

THEOREM 2.3. *Let $\tau = (Q_t)$ be a stopping time. Then*

(i) *$P_{\tau(T)}$ is a self-adjoint projection on $\mathscr{H}$ for any $T \in \mathscr{P}$;*

(ii) *if $T_1, T_2 \in \mathscr{P}$ with T_2 finer than T_1, then $P_{\tau(T_1)} \geqslant P_{\tau(T_2)}$; and*

(iii) *if $\sigma = (Q'_t)$ is another stopping time with $\sigma \geqslant \tau$, then $P_{\sigma(T)} \geqslant P_{\tau(T)}$ for any $T \in \mathscr{P}$.*

Proof. (i) $P_{\tau(T)}=\sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i}$. By 1.2 each $\Delta Q_{t_i} P_{t_i}$ is a projection and the family $\{\Delta Q_{t_i} P_{t_i} : i=1, ..., n\}$ is disjoint. Hence the result follows.

(ii) Suppose $T_2 = T_1 \cup \{q\}$, where $T_1 = \{t_0, ..., t_n\}$ with $t_i < t_{i+1}$ and $q \in (t_r, t_{r+1})$, $r+1<n$. Then for $\xi \in \mathscr{H}$,

$$\begin{aligned}P_{\tau(T_1)} \circ P_{\tau(T_2)}(\xi) = &\sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} \left(\sum_{i=1}^{r} \Delta Q_{t_i} P_{t_i}(\xi) \right) \\ &+ \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} (Q_q - Q_{t_r}) P_q(\xi) \\ &+ \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} (Q_{t_{r+1}} - Q_q) P_{t_{r+1}}(\xi) \\ &+ \sum_{j=1}^{n} \Delta Q_{t_j} P_{t_j} \left(\sum_{i=r+2}^{n} \Delta Q_{t_i} P_{t_i}(\xi) \right).\end{aligned}$$

Using 1.2 and the orthogonality of ΔQ_{t_i} gives the result. The general case follows similarly.

(iii) Let $T \in \mathscr{P}$, say $T=\{t_0, ..., t_n\}$. Then

$$I = \sum_{i=1}^{n} \Delta Q_{t_i} = \sum_{i=1}^{n} \Delta Q'_{t_i}.$$

Now $P_{\tau(T)} \circ P_{\sigma(T)} = \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i} (\sum_{j=1}^{n} \Delta Q'_{t_j} P_{t_j})$. But $\Delta Q_{t_i} P_{t_i} (\sum_{j=1}^{n} \Delta Q'_{t_j} P_{t_j}) = \sum_{j=i}^{n} \Delta Q_{t_i} P_{t_i} \Delta Q'_{t_j}$, by 1.2 and observing that $\Delta Q'_{t_j} \leqslant Q_{t_{i-1}}$ if $j \leqslant i-1$. Hence

$$\begin{aligned}P_{\tau(T)} \circ P_{\sigma(T)} &= \sum_{i=1}^{n} \sum_{j=i}^{n} \Delta Q_{t_i} P_{t_i} \Delta Q'_{t_j} \\ &= \sum_{i=1}^{n} \left(\sum_{j=i}^{n-1} \Delta Q_{t_i} P_{t_i} \Delta Q'_{t_j} + \Delta Q_{t_i} P_{t_i} \left(I - \sum_{k=1}^{n-1} \Delta Q'_{t_k} \right) \right) \\ &= \sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i} \\ &= P_{\tau(T)}.\end{aligned}$$

Hence $P_{\tau(T)} \leqslant P_{\sigma(t)}$.

DEFINITION 2.4. For a stopping time $\tau = (Q_t)$ we define the time projection at τ, P_τ, as

$$P_\tau = \inf_{T \in \mathscr{P}} \sum_i \Delta Q_{t_i} P_{t_i}.$$

For any simple $\mathscr{H}$-valued martingale $\eta = (P_t(\xi))$ we define the stopped martingale η_τ by

$$\eta_\tau = P_\tau \xi.$$

Let $\tilde{t}$ be the stopping time given by

$$\tilde{t}(s) = \begin{cases} 0 & \text{if} \quad s \leqslant t \\ I & \text{if} \quad t > s, \end{cases}$$

then we have the following consistency lemma:

LEMMA 2.5. $\eta_t = \eta_{\tilde{t}}$.

THEOREM 2.6 (Optional Stopping Theorem). *Let τ and σ be two stopping times with $\sigma \geqslant \tau$ and let $\eta = (\eta_t)$ be a simple $\mathscr{H}$-valued martingale. Then*

$$P_\tau(\eta_\sigma) = \eta_\tau.$$

Proof. Since η is a simple martingale,

$$\eta_t = P_t \xi \qquad \text{for some} \quad \xi \in \mathscr{H}.$$

From 2.3(iii), it is clear that

$$P_\tau \leqslant P_\sigma.$$

Now

$$\begin{aligned} \eta_\tau &= P_\tau \xi \\ &= P_\tau \circ P_\sigma \xi \\ &= P_\tau \eta_\sigma. \end{aligned}$$

3. A RANDOM STOPPING THEOREM

We now specialise to the quasi-free representation of the CAR algebra over a Hilbert space $\mathscr{K}$. In the context of Section 1, we identify $\mathscr{K}$ with $L^2(\mathbb{R}^+)$ and $\mathscr{A}$ with the unital C^*-algebra generated by $\{b^*(u), b(u) : u \in \mathscr{K}\}$ satisfying

$$b(f)b^*(g) + b^*(g)b(f) = \langle f, g \rangle I$$

$$b(f)b(g) + b(g)b(f) = 0$$

$$b^*(\lambda f + g) = \lambda b^*(f) + b^*(g)$$

$$\|b(f)\| = \|b^*(f)\| = \|f\|$$

for all $f, g \in \mathscr{K}$ and $\lambda \in \mathbb{C}$.

Let $R \in B(\mathscr{K})$ with $0 \leqslant R \leqslant I$ and w be the gauge invariant quasi-free state on $\mathscr{A}$ determined by $w(b^*(f)b(g)) = \langle f, Rg \rangle$. Let $(\mathscr{H}, \pi, \Omega)$ be the GNS triple associated with $\mathscr{A}$ [4]. For each $t \in \mathbb{R}^+$, let $\mathscr{K}_t = L^2([0, t])$ and $\mathscr{A}_t$ be the CAR algebra over $\mathscr{K}_t$ and $(\mathscr{H}_t, \pi, \Omega)$ be the GNS triple of $\mathscr{A}$. Then there exists a family of conditional expectations (M_t) satisfying the properties listed in Section 1, which is given by Evans [6]:

$$M_t(b(f)) = M_t(b(f_t \oplus f')) = b(f_t) + \langle \Omega, b(f')\Omega \rangle I,$$

where $f_t \in \mathscr{K}_t$ and $f' \in \mathscr{K}_t^{\perp}$ so that $f = f_t \oplus f'$.

Let $u \in \mathscr{K}$, then $X_s = \lambda_1 b^*(\chi_{[0,s]}u) + \lambda_2 b(\chi_{[0,s]}u)$ defines an $\mathscr{A}$-valued martingale for any $\lambda_1, \lambda_2 \in \mathbb{C}$. Thus we can define stochastic integration with respect to (X_t) as in [3, 7]. Thus if $h : \mathbb{R} \to \mathscr{A}$ is a simple process, that is, for each t, $h(t) \in \mathscr{A}_t$ and

$$h(t) = \sum_i h_{i-1} \chi_{[t_{i-1}, t_i)}(t),$$

then

$$\int_0^\infty dX_s h(s) = \sum_i \Delta X_{t_i} h_{i-1} \Omega.$$

Then this integral obeys the isometry property [3, 8]

$$\left\| \int_0^\infty dX_s\, h(s) \right\|^2 = \int_0^\infty \|h(s)\Omega\|^2 \, d\mu(s),$$

where $d\mu(s) = (|\lambda_1|^2 (1 - \rho(s)) + |\lambda_2|^2 \rho(s)\, |u(s)|^2)\, ds$ and ρ is the multiplication operator on $\mathscr{K}$ corresponding to $R \in B(\mathscr{K})$ [3, 8]. This extends to any process in $L^2(\mathbb{R}^+, \mu, \mathscr{H})$.

THEOREM 3.1 (A Random Stopping Theorem). *Let $\eta_t = \int_0^\infty f(s)\, dX_s$ define an $\mathscr{H}$-valued martingale* [3], *where f is an element of $L^2(\mathbb{R}^+, \mu, \mathscr{H})$ and adapted to the filtration $(\mathscr{H}_t)$, and suppose η_∞ exists. Let $\tau_n = (Q_t^n)$ be a sequence of stopping time converging pointwise strongly to the stopping time $\tau = (Q_t)$, i.e., for each $t \in \mathbb{R}^+$, $\tau_n(t) \to \tau(t)$ strongly. Then $\eta_{\tau_n} \to \eta_\tau$.*

Before we can prove the theorem we need the following lemmas:

LEMMA 3.2 [2]. *Let $\tau = (Q_t)$ be any stopping time. Then there exists a sequence (τ_n) of simple stopping time converging to τ pointwise strongly. Explicitly, there exists a sequence (T_n) of partitions of $[0, \infty]$ such that for each $s \in \mathbb{R}^+$,*

$$Q_s^{T_n} \to Q_s \qquad \textit{strongly},$$

where

$$Q_s^{T_n} = \sum_{i=1}^{m} Q_{t_{i-1}^n} \chi_{[t_{i-1}^n, t_i^n)}(s)$$

and

$$T_n = \{t_0^n, ..., t_m^n\} \in \mathscr{P}.$$

LEMMA 3.3. *Let η be as in the statement of the theorem and $\tau = (Q_t)$ be any stopping time. Then*

(i) $\int Q_s f(s)\, dX_s$ *is a well-defined stochastic integral.*

(ii) *Let $T = \{t_0, ..., t_n\}$ be any partition of $[0, \infty]$, then*

$$\eta_{\tau(t)} \equiv P_{\tau(T)} \eta_\infty = \int_0^\infty (1 - Q_s^T) f(s)\, dX_s.$$

(iii) $P_{\tau(T)} \int_0^\infty (1 - Q_s) f(s)\, dX_s = \int_0^\infty (1 - Q_s) f(s)\, dX_s.$

(iv) $\eta_\tau = \int_0^\infty (1 - Q_s) f(s)\, dX_s.$

Proof. (i) Let $h\colon \mathbb{R}^+ \to \mathscr{H}$ be given by $h(s) = Q_s f(s)$. Then by 1.2, $P_s h(s) = Q_s P_s f(s) = Q_s f(s) = h(s)$, hence h is adapted. Also

$$\|h\|^2 = \int_0^\infty \|Q_s f(s)\|^2\, d\mu(s)$$
$$\leqslant \int_0^\infty \|f(s)\|^2\, d\mu(s) < \infty.$$

To show that h is measurable [5], it is enough to consider f elementary. The general case follows by linearity and continuity. Suppose $f(s) = f_0$ and let (τ_n) be a sequence of simple stopping time approximating τ [2], then

$$\int_0^\infty \|\tau_n(s) f_0 - \tau(s) f_0\|^2\, d\mu(s) = \int_0^\infty \|(Q_s^n - Q_s) f_0\|^2\, d\mu(s).$$

But $\|(Q_s^n - Q_s) f_0\|^2 \to 0$ as $n \to \infty$ and $\|(Q_s^n - Q_s) f_0\|^2 \leqslant 2\, \|f_0\|^2$, f is in L^2, and furthermore $d\mu(s)$ is a finite measure. Hence the dominated convergence theorem is applicable and

$$\tau_n(s) f_0 \to \tau(s) f \qquad \text{in } L^2,$$

i.e.,

$$Q_s^n f_0 \to Q_s f_0 \qquad \text{in } L^2.$$

But $h_n(s) = Q_s^n f_0$ is clearly a sequence of simple L^2 processes, hence h is measurable. Hence $\int_0^\infty h(s)\, dX_s$ is a well-defined stochastic integral [3].

(ii) Since η_∞ exists, η is a simple martingale [1], i.e., $\eta_t = P_t \eta_\infty$, so that

$$\begin{aligned}\eta_{\tau(T)} = P_{\tau(T)}\eta_\infty &= \left(\sum_{i=1}^{n} \Delta Q_{t_i} P_{t_i}\right)\eta_\infty \\ &= \left(I = \sum_{i=1}^{n} Q_{t_{i-1}} \Delta P_{t_i}\right)\eta_\infty \\ &= \int_0^\infty f(s)\, dX_s - \sum_i Q_{t_{i-1}} \int_{t_{i-1}}^{t_i} f(s)\, dX_s.\end{aligned}$$

Now

$$Q_{t_{i-1}} \int_{t_{i-1}}^{t_i} f(s)\, dX_s = Q_{t_{i-1}} \operatorname{Lim}_n \sum_j \Delta X_{s_j} \Gamma f_{j-1}^n,$$

where Γ is a unitary operator satisfying $\Gamma\Omega = \Omega$, $A\, \Delta X_{t_i} = \Delta X_{t_i} \Gamma A \Gamma^{-1}$ [3], (f^n) is a sequence of simple processes converging to f in L^2. Since $Q_{t_{i-1}}$ is continuous and linear, the above is $\operatorname{Lim}_n \sum \Delta X_{s_j} \Gamma Q_{t_{i-1}} f_{j-1}^n$. But $Q_{t_{i-1}} f^n(s) \to Q_{t_{i-1}} f(s)$ and $Q_{t_{i-1}} f$ is measurable from (i), hence the above becomes $\int_{t_{i-1}}^{t_i} Q_{t_{i-1}} f(s)\, dX_s$. Thus $\Sigma_i Q_{t_{i-1}} \int_{t_{i-1}}^{t_i} f(s)\, dX_s = \int_0^\infty Q_s^T f(s)\, dX_s$ and $\eta_{\tau(T)} = \int_0^\infty (1 - Q_s^T) f(s)\, dX_s$.

(iii) Let $Z = \int_0^\infty (1 - Q_s) f(s)\, dX_s$. Then

$$P_{\tau(T)} Z = \int_0^\infty (1 - Q_s^T)(1 - Q_s) f(s)\, dX_s.$$

But by construction $Q_s^T = Q_{t_{i-1}}$ if $s \in [t_{i-1}, t_i)$. Hence $Q_s^T \leqslant Q_s$, so that $(1 - Q_s^T)(1 - Q_s) = 1 - Q_s$ and the result follows.

(iv) Let T_n be any partition of $[0, \infty]$ coarser than T, so that $P_{\tau(T)} \circ P_{\tau(T_n)} = P_{\tau(T)}$. Then

$$\begin{aligned}\|Z - P_{\tau(T)}\eta_\infty\|^2 &= \|P_{\tau(T)}(Z - P_{\tau(T_n)}\eta_\infty)\|^2 \\ &\leqslant \|Z - P_{\tau(T_n)}\eta_\infty\|^2 \\ &= \left\| \int_0^\infty (Q_s - Q_s^{T_n}) f(s)\, dX_s \right\|^2 \\ &= \int_0^\infty \|(Q_s - Q_s^{T_n}) f(s)\|^2\, d\mu(s).\end{aligned}$$

Now let (T_n) be a sequence of partitions approximating τ, so that $\|(Q_s - Q_s^{T_n}) f(s)\|^2 \to 0$ for each $s \in \mathbb{R}^+$ and $\|(Q_s - Q_s^{T_n}) f(s)\|^2 \leqslant 2 \|f(s)\|^2$. Hence by the dominated convergence theorem

$$\int_0^\infty \|(Q_s - Q_s^{T_n}) f(s)\|^2 \, d\mu(s) \to 0.$$

That is, given $\varepsilon > 0$, there is an N_ε such that $\forall n \geqslant N_\varepsilon$,

$$\|Z - P_{\tau(T_n)} \eta_\infty\|^2 < \varepsilon^2.$$

Now set $S = T_{N_\varepsilon}$. Then $\forall T' \supseteq S$

$$\|Z - P_{\tau(T')} \eta_\infty\|^2 < \|Z - P_{\tau(T_{N_\varepsilon})} \eta_\infty\|^2 < \varepsilon^2.$$

Hence

$$Z = P_\tau \eta_\infty.$$

Proof of 3.1. From above we know that

$$\eta_\tau = \int_0^\infty (1 - Q_s) f(s) \, dX_s$$

and

$$\eta_{\tau_n} = \int_0^\infty (1 - Q_s^n) f(s) \, dX_s.$$

Thus

$$\|\eta_\tau - \eta_{\tau_n}\|^2 = \int_0^\infty \|(Q_s - Q_s^n) f(s)\|^2 \, d\mu(s).$$

As $Q_s^n \to Q_s$ strongly and $\|(Q_s - Q_s^n) f(s)\|^2 \leqslant 2 \|f(s)\|^2$, the dominated convergence theorem applies. Hence the result.

THEOREM 3.4. *Let* (τ_n) *and* τ *be as in Theorem* 3.1. *Let* J *be a finite interval in* $\mathbb{R}^+$ *and* ρ *be bounded away from* 0 *and* 1 *on* J. *Let* $\eta = (\eta_t)$ *be any* $\mathscr{H}$*-valued,* L^2*-martingale on* J. *Then* $\eta_{\tau_n} \to \eta_\tau$.

Proof. For such ρ, there exists a martingale representation theorem [8]

$$\eta_t = \int_0^t f(s) \, db_s^* + \int_0^t g(s) \, db_s,$$

where $f, g \in L^2(J, d\mu, \mathscr{H})$. Hence the arguments of Theorem 3.1 are applicable.

Remark. When dealing with the CCR algebra [4], there exists a martingale representation theorem [7] and once again we can prove the random convergence theorem for this model.

REFERENCES

1. C. BARNETT, Supermartingales on semi-finite Von Neumann algebras, *J. London Math. Soc. (2)* **24** (1981), 175–181.
2. C. BARNETT AND T. J. LYONS, Stopping non-commutative processes, *Math. Proc. Cambridge Philos. Soc.* (1985).
3. C. BARNETT, R. F. STREATER, AND I. F. WILDE, Quasi-free quantum stochastic integrals for the C.A.R. and C.C.R., *J. Funct. Anal.* **52** (1983), 19–47.
4. O. BRATELLI, AND D. ROBINSON, "Operator Algebras and Quantum Statistical Mechanics," Vol. 2, Springer-Verlag, Heidelberg/Berlin/New York, 1981.
5. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Part I, Wiley, New York, 1958.
6. P. E. EVANS, Completely positive quasi-free map of the C.A.R. algebra, *Comm. Math. Phys.* **70** (1979), 53–68.
7. R. L. HUDSON AND J. M. LINDSAY, A non-commutative martingale representation theorem for non-fock quantum Brownian motion, *J. Funct. Anal.* **61** (1985), 202–221.
8. J. M. LINDSAY, Fermion martingales, *Z. Wahrsch. Verw. Gebiete*, in press.